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The Effects of Quantization on Signal Processing

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ABSTRACT

Typically an analog signal from a space system is sampled, quantized by Analog-to-Digital (A/D) conversion, merged into a bit stream, communicated to a ground station, received by the ground station, and processed by the ground station to extract useful information for dissemination to the users. The cost of each of these steps is reduced as the number of quantization steps is reduced in the A/D converter. The number of quantization steps should be as small as possible without losing the required information content. This report deals specifically with the accuracy of averages as a function of the number of quantized samples used to compute the averages with the noise on the analog signal as a parameter. For example, the success of the Visible Infrared Spin Scan Radiometer (VISSR) Atmospheric Sounder (VAS) Demonstration depends upon temporally averaging multiple samples in an effort to reduce noise to a sufficiently low level such that temperature profile sounding is made possible. A tutorial description of this process is presented which demonstrates the viability and limitation of noise reduction by averaging quantized samples of an analog signal.

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ILLUSTRATIONS

TABLE

LIST OF SYMBOLS

A = The average of N samples of an analog signal, where the i^{th} sample is S_i.

A_{Qjk} = The average of N quantized samples of an analog signal, where the i^{th} analog sample is S_{ijk}.

a = The mean of a gaussian distribution.

$F(x)$ = This function is defined by equation D-3.

L = The number of averages A_{Qjk} which were computed to estimate the variance V_{Qj} about the true value T_j.

M = The mean value of the set of averages, where each average is determined from N samples of an analog signal.

M_Q = The mean value of samples of an analog signal which are uniformly distributed in quantization step Q, where $M_Q = Q$.

N = The number of samples used to compute averages.

$p(y)$ = The gaussian distribution with mean $a = 0$ and standard deviation σ_s.

$P(z)$ = The gaussian distribution with mean $a = T$ and standard deviation σ_s.

Q = An integer which is called the quantized sample value of an analog signal. If a sample of an analog signal has a value between $Q - ½$ and $Q + ½$, then its quantized value is Q.

Q_{ijk} = The value of Q for sample S_{ijk} of an analog signal.

$R = 5$ = The number of true values T_j for which V_{Qj}($j = 1$ to R) was computed.

S = An analog signal.

S_i = The i^{th} sample of an analog signal.

S_{ijk} = The i^{th} sample of an analog signal which has T_j for a true value and is used to compute the k^{th} average A_{Qjk}.

T = The true value of an analog signal.

T_j = The j^{th} true value ($j = 1$ to 5) chosen in equal increments between 0 and .5. ($T_1 = .05, T_2 = .15, T_3 = .25, T_4 = .35, T_5 = .45$.)

U = A uniformly distributed random number between 0 and 1.

V_A = The variance of the set of averages A about the mean $M = T$.

V_{QN} = The average variance of the set of V_{Qj}, $j = 1$ to R.

V_{QNj} = The variance of the set of averages, A_{Qjk}, for each particular true value T_j.

x = $T - Q$.

y = A random variable with a gaussian distribution.

y_i = The i^{th} value of $y(i = 1$ to $w + 1)$ such that the area under the gaussian distribution $p(y)$ between y_i and y_{i+1} is constant ($= .5/w$). We take $0 = y_1, < y_2 < \ldots < y_{w+1}$.

$\overline{y}_i$ = $\dfrac{y_i + y_{i+1}}{2}$, where $i = 1$ to w.

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

η = Gaussian noise with zero mean and standard deviation σ_s.

η_i = The i^{th} sample of η.

η_{ik} = The i^{th} sample of η used to compute the k^{th} average A_{Qjk}.

σ = The standard deviation of η.

σ_A = The standard deviation of the set of averages A about the mean $M = T$.

$\sigma_Q = \dfrac{1}{\sqrt{12}}$ = The rms error incurred by quantizing a sample of a clean (noise free) signal.

σ_{QNj} = The standard deviation of the set of averages A_j about the true value T_j.

σ_{QN} = The standard deviation associated with V_{QN}.

$\sigma_{Q\infty}$ = The value of σ_{QN} for very large $N(N = \infty)$.

σ_s = The standard deviation of the noise per sample of an analog signal.

Operators

$E\{g(t)\}$ = $\int_{-\infty}^{\infty} g(\chi)p(\chi)d\chi$

$\mathrm{Int}(\chi)$ = The largest integer which is less than or equal to χ.

$\mathrm{sgn}(\chi)$ = $\begin{cases} +1 \text{ for } \chi \geq 0 \\ -1 \text{ for } \chi < 0 \end{cases}$

THE EFFECTS OF QUANTIZATION ON SIGNAL PROCESSING

Averages computed from N samples of an analog signal have a standard deviation σ_A about the true signal given by (see Appendix A)

$$\sigma_A = \frac{\sigma_s}{\sqrt{N}} \tag{1}$$

where

σ_s = The standard deviation of the noise per sample of an analog signal.

N = The number of samples of an analog signal which are used to compute the average.

σ_A = The standard deviation of the set of averages. Each average is determined from N samples of an analog signal.

Equation 1 does not include the effects of quantization errors. Quantization of a perfectly clean sample (noise free) of an analog signal introduces a root mean square (rms) quantization noise of (see Appendix B):

$$\begin{bmatrix}\text{rms quantization noise} \\ \text{of a sample of a clean} \\ \text{signal}\end{bmatrix} = \frac{1}{\sqrt{12}} \text{ quantization steps} \tag{2}$$

Both noise on the analog signal and quantization errors degrade the accuracy of the average of N samples. Figure 1 presents the combined effect on the accuracy of the average. This figure gives σ_{QN} (the standard deviation of the averages) as a function of N (the number of quantized samples of an analog signal used to compute the average) for various values of σ_s (the standard deviation of the noise per sample of the analog signal). The method of computation is described in Appendix C. Table 1 presents the asymptotic values of $\sigma_{QN}(\sigma_{Q\infty})$ as a function of σ_s for very large values of $N(N \to \infty)$. See Appendix D. It should be noted that the curve for $\sigma_s = .2$ in Figure 1 is asymptotic to $\sigma_{Q\infty} = .1023$ as is indicated in Table 1.

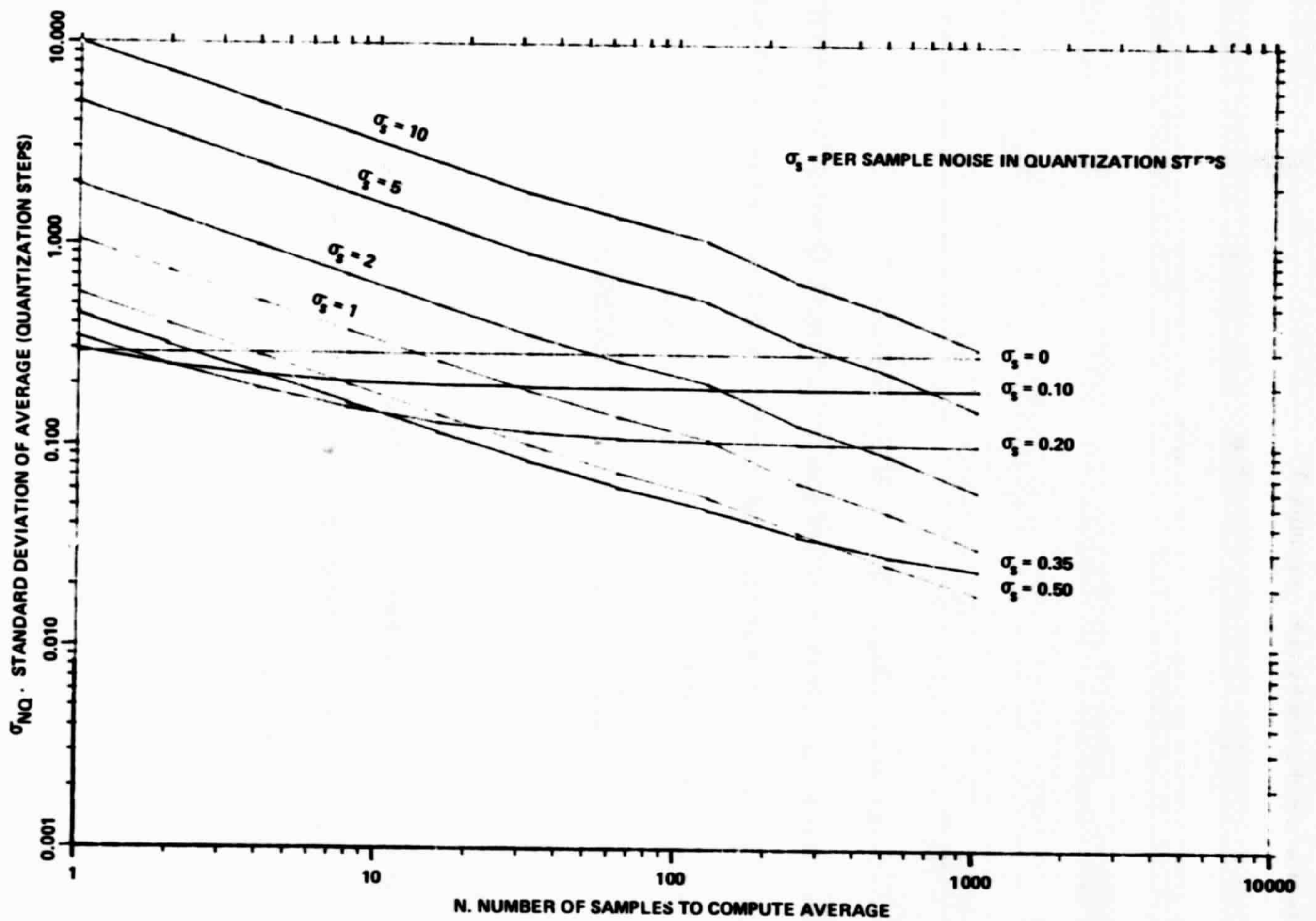

Figure 1. Standard Deviation of the Average as a Function of the Number of Quantized Samples Used to Compute the Average

Table 1

Asymptotic Values $\sigma_{QN_{(N=\infty)}}$ as a Function of σ_s

σ_s (quantization steps)	$\sigma_{Q\infty}$ (quantization steps)
0	0.288675134 = $1/\sqrt{12}$
0.100000000	0.192150011
0.200000000	0.102308167
0.300000000	0.038089263
0.400000000	0.009565700
0.500000000	0.001618753
0.600000000	0.000184583
0.700000000	0.000014182
0.800000000	0.000000734
0.900000000	0.000000026
1.000000000	0.000000001
1.100000000	0.000000000
$\sigma_s > 1.1$	0

The set of averages where each average is determined from N quantized samples has a standard deviation about the true analog signal of σ_{QN}. If the standard deviation of the noise of the analog signal is σ_s, then

$$\sigma_{QN} = \frac{1}{\sqrt{12}} \text{ for } \sigma_s = 0$$

$$\frac{\sigma_s}{\sqrt{N}} < \sigma_{QN} < \frac{1}{\sqrt{12}} \text{ for } 0 < \sigma_s < 1, N \geqslant 12$$

and

$$\sigma_{QN} = \frac{\sigma_s}{\sqrt{N}} \text{ for } \sigma_s \geqslant 1$$

where σ_s and σ_{QN} are measured in unit quantization steps. For $0 < \sigma_S < 1$ and $N < 12$ no generalization can be made with regard to the value σ_{QN}, however Figure 1, does give specific results for σ_s = 0.1, 0.2, 0.35 and 0.5.

APPENDIX A

VARIANCE OF THE ANALOG AVERAGE

Let S_i be the i^{th} sample of an analog signal whose true value is T, then S_i is given by

$$S_i = T + \eta_i \tag{A-1}$$

where

T = The true signal = a constant

η_i = Gaussian noise with zero mean and standard deviation σ_s

The average A of N samples is computed from the expression

$$A = \frac{1}{N} \sum_{i=1}^{N} S_i \tag{A-2}$$

Substituting Eqn. (A-1) into Eqn. (A-2) the following result is obtained

$$A = \frac{1}{N} \sum_{i=1}^{N} (T + \eta_i)$$

$$A = T + \frac{1}{N} \sum_{i=1}^{N} \eta_i \tag{A-3}$$

The mean M of the average A is computed as follows:

$$M = E\{A\} = E\left\{T + \frac{1}{N} \sum_{i=1}^{N} \eta_i\right\} \tag{A-4}$$

$$= E\{T\} + \frac{1}{N} \sum_{i=1}^{N} E\{\eta_i\}$$

or

$$M = T + \frac{1}{N} \sum_{i=1}^{N} E\{\eta_i\} \tag{A-5}$$

Since η_i has mean zero, i.e.

$$E(\eta_i) = 0, \tag{A-6}$$

Eqn. (A-5) becomes

$$M = T \tag{A-7}$$

Next compute the variance, V_A, of the average A about the mean M as follows

$$V_A = E\left\{(A - M)^2\right\} \tag{A-8}$$

Substituting Eqns. (A-3) and (A-7) into Eqn. (A-8) results in

$$V_A = E\left\{\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \eta_i\eta_j\right\}$$

or

$$V_A = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E\left\{\eta_i\eta_j\right\} \tag{A-9}$$

By assuming that η_i and η_j are independent for $i \neq j$, and utilizing the definition of the expectation operator, the following result is obtained

$$E\left\{\eta_i\eta_j\right\} = \int_{-\infty}^{\infty} \eta_i\eta_j\, p(\eta_i)d\eta_i$$

$$= \sigma_s^2\,\delta_{ij} \tag{A-10}$$

where the Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases} \tag{A-11}$$

Substitute Eqn. (A-10) into Eqn. (A-9) to obtain

$$V_A = \frac{\sigma_s^2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{ij} \tag{A-12}$$

But

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \delta_{ij} = N \tag{A-13}$$

Substitution of Eqn. (A-13) into Eqn. (A-12) gives the variance of the average

$$V_A = \frac{\sigma_s^2}{N} \tag{A-14}$$

The standard deviation of the average σ_A is then given by

$$\sigma_A = \frac{\sigma_s}{\sqrt{N}} \tag{A-15}$$

APPENDIX B

SINGLE SAMPLE QUANTIZING ERRORS

Given an analog signal S (measured in quantization step units) which is to be quantized. Without loss of generality, it may be stated that the value of S falls in the integer quantization step Q. In other words, S lies anywhere between Q – ½ and Q + ½ with equal probability. Given this constraint, the mean value of S, M_Q, is given by

$$M_Q = \int_{Q-\frac{1}{2}}^{Q+\frac{1}{2}} S dS = \frac{S^2}{2} \Bigg|_{Q-\frac{1}{2}}^{Q+\frac{1}{2}}$$

or

$$M_Q = Q \qquad \text{(B-1)}$$

If S falls in the integer quantization step Q, the value of S is approximated by Q and the error S – Q, for one sample, lies between –½ and +½.

Approximating S by Q over a whole population of S's, the root mean square (rms) error σ_Q in the estimate of S is given by

$$\sigma_Q = \left[\int_{Q-\frac{1}{2}}^{Q+\frac{1}{2}} (S - Q)^2 dS \right]^{\frac{1}{2}}$$

$$\sigma_Q = \frac{1}{\sqrt{12}} \text{ quantization steps} \qquad \text{(B-2)}$$

APPENDIX C

VARIANCE OF THE AVERAGE WITH QUANTIZING

The quantized signal Q_{ijk} is computed from the expression

$$Q_{ijk} = \text{Int}\,(S_{ijk} + .5) \tag{C-1}$$

where

Int (χ) = The largest integer which is less than or equal to χ.

The i^{th} sample of an analog signal which has T_j for a true value and gaussian noise η_{ik}, and is used to compute the k^{th} average A_{Qjk} is computed from

$$S_{ijk} = T_j + \eta_{ik} \tag{C-2}$$

Appendix E shows a method of computing gaussian noise η.

The average A_{Qjk} of N quantized samples is computed from the expression

$$A_{Qjk} = \frac{1}{N} \sum_{i=1}^{N} Q_{ijk} \tag{C-3}$$

To determine how well A_{Qjk} approximates the true value T_j, compute the variance V_{QNj} of A_{Qjk} about T_j as follows:

$$V_{QNj} = \frac{1}{(L-1)} \sum_{k=1}^{L} (A_{Qjk} - T_j)^2 \tag{C-4}$$

The standard deviation σ_{QNj} of A_{Qjk} is given by

$$\sigma_{QNj} = \sqrt{V_{QNj}} \tag{C-5}$$

Figures C-1 through C-4 present σ_{QNj} for T_j = 0, 0.01, 0.25 and 0.5 respectively. These figures demonstrate that for $\sigma_s > 1$ the values of σ_{QNj} are independent of the value of T_j, and σ_{QNj} is a function of T_j for $\sigma_s < 1$. For $T_j = 0$ (the middle of the quantization step) Figure C-1 indicates that $\sigma_{QNj} = 0$ for $\sigma_s < 0.14$. Figure C-2 shows that for $T_j = 0.01$ (0.01 units from the middle of the step), $\sigma_{QNj} = 0.01$ for $\sigma_s < 0.14$. Figure C-3 shows that for $T_j = 0.25$, $\sigma_{QNj} = 0.25$ for $\sigma_s < 0.07$. For $T_j = 0.5$ (true value on the threshold between quantization steps 0 and 1), the curves σ_{QNj} versus N are identical for $\sigma_s < 0.2$.

It is necessary to obtain the average variance for equally likely values of T_j. Thus taking R values of T_j spaced uniformly between 0 and 0.5 results in

$$V_{QN} = \frac{1}{R} \sum_{j=1}^{R} V_{QNj} \qquad \text{(C-6)}$$

The standard deviation σ_{QN} of the average of N quantized samples of an analog signal about the true signal which is uniformly distributed over a quantization step is given by

$$\sigma_{QN} = \sqrt{V_{QN}} \qquad \text{(C-7)}$$

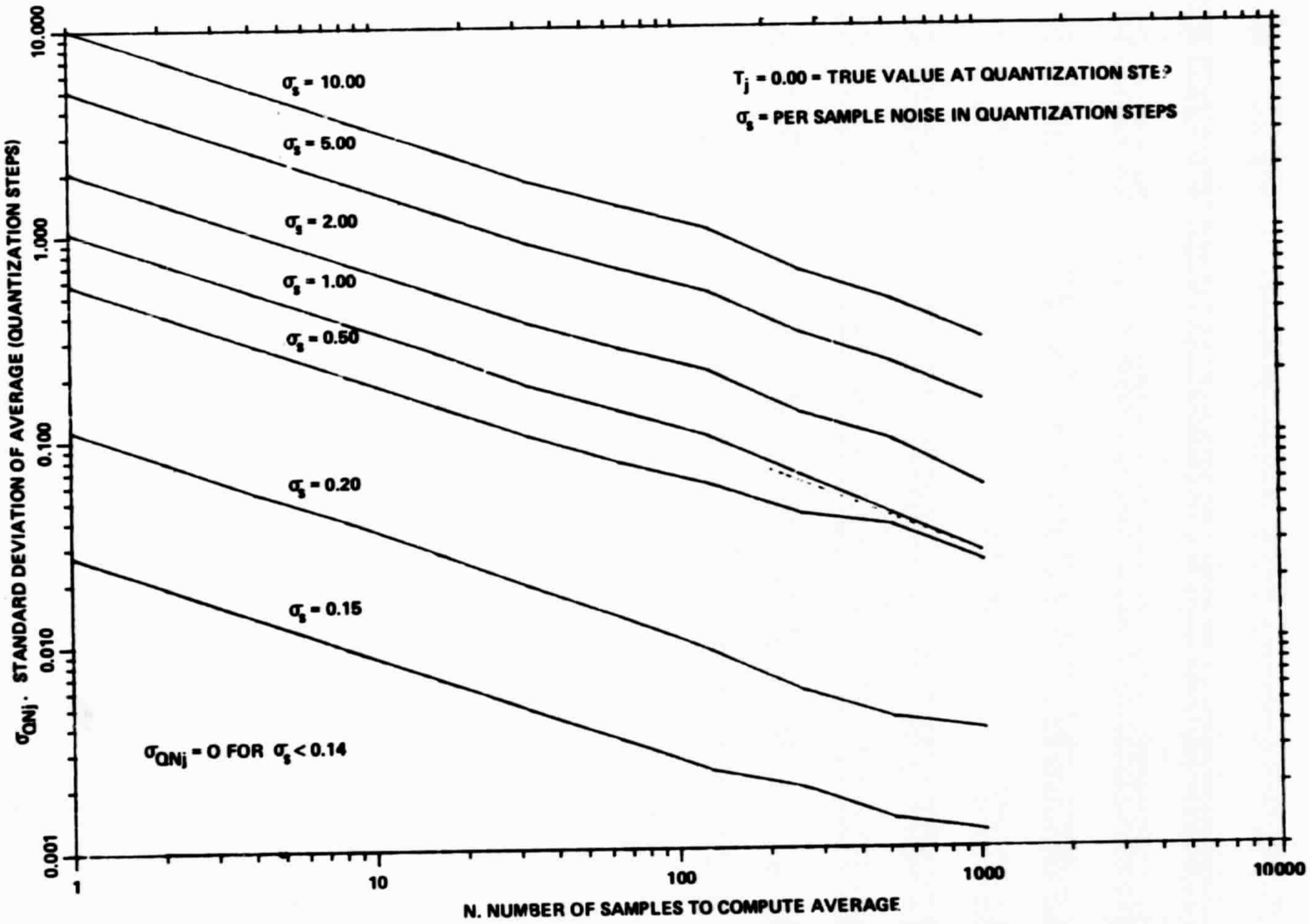

Figure C-1. Standard Deviation of the Average as a Function of the Number of Quantized Samples Used to Compute the Average for a True Signal Value $T_j = 0$

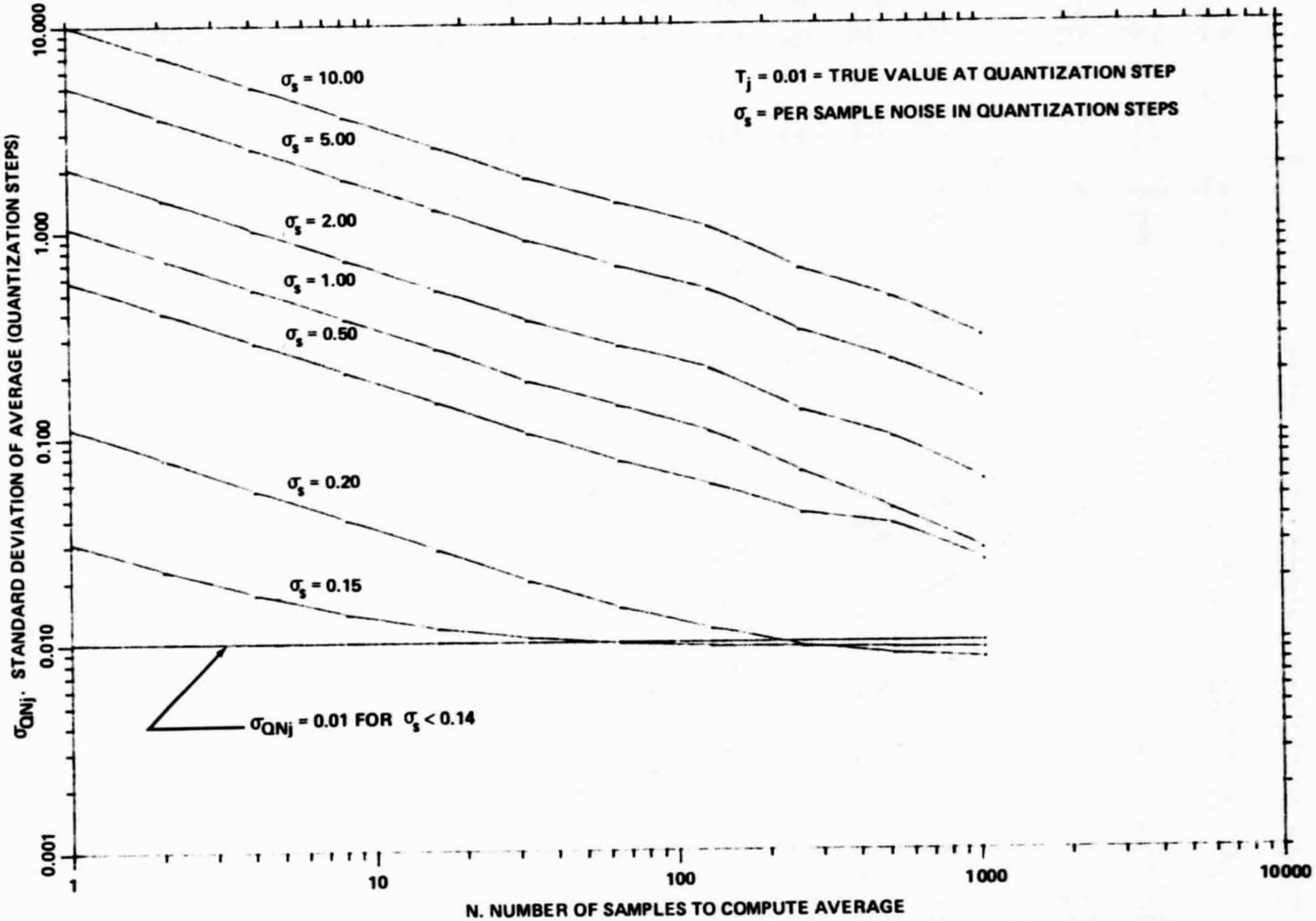

Figure C-2. Standard Deviation of the Average as a Function of the Number of Quantized Samples Used to Compute the Average for a True Signal Value $T_j = 0.01$

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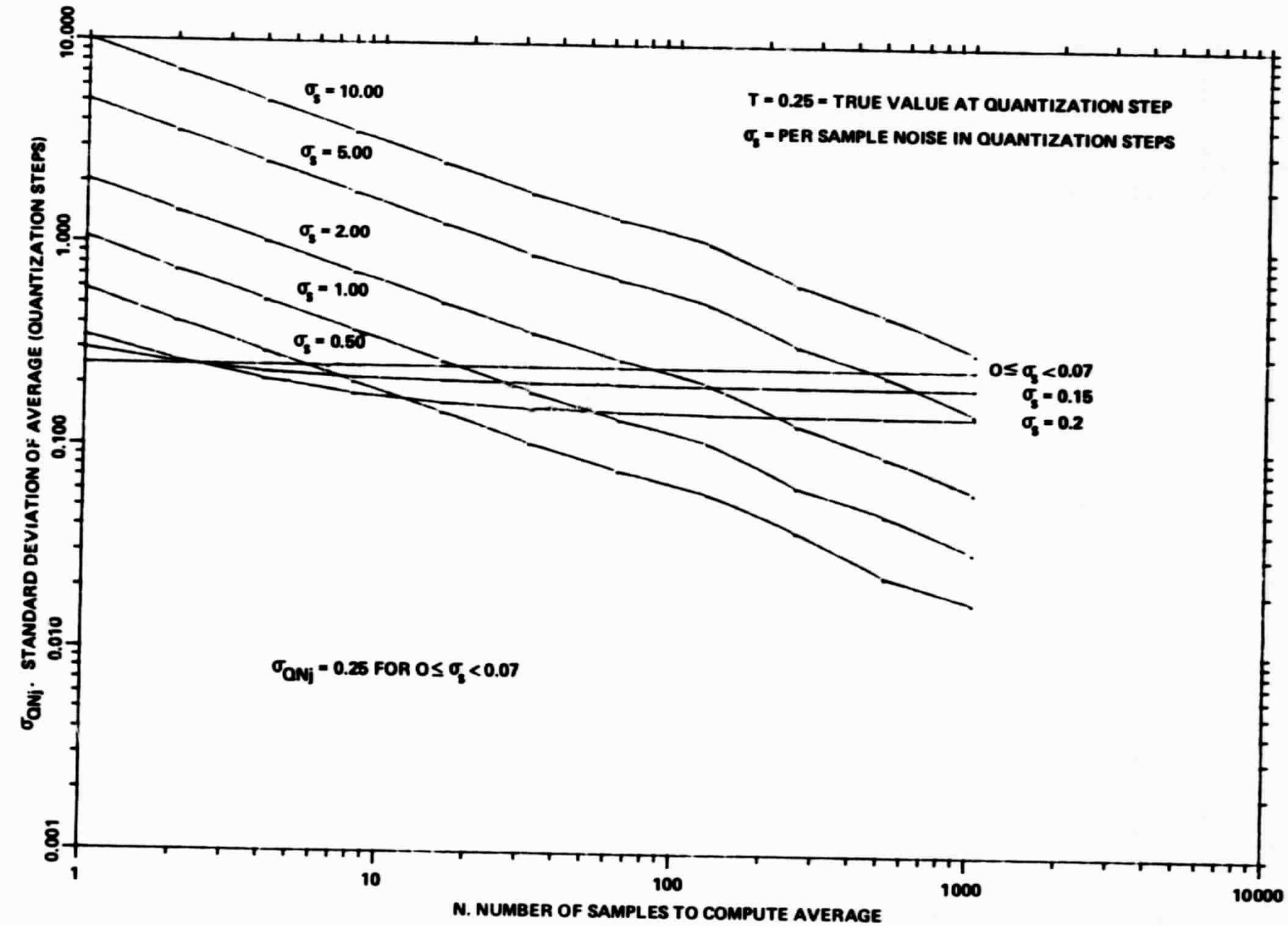

Figure C-3. Standard Deviation of the Average as a Function of the Number of Quantized Samples Used to Compute the Average for a True Signal Value $T_j = 0.25$

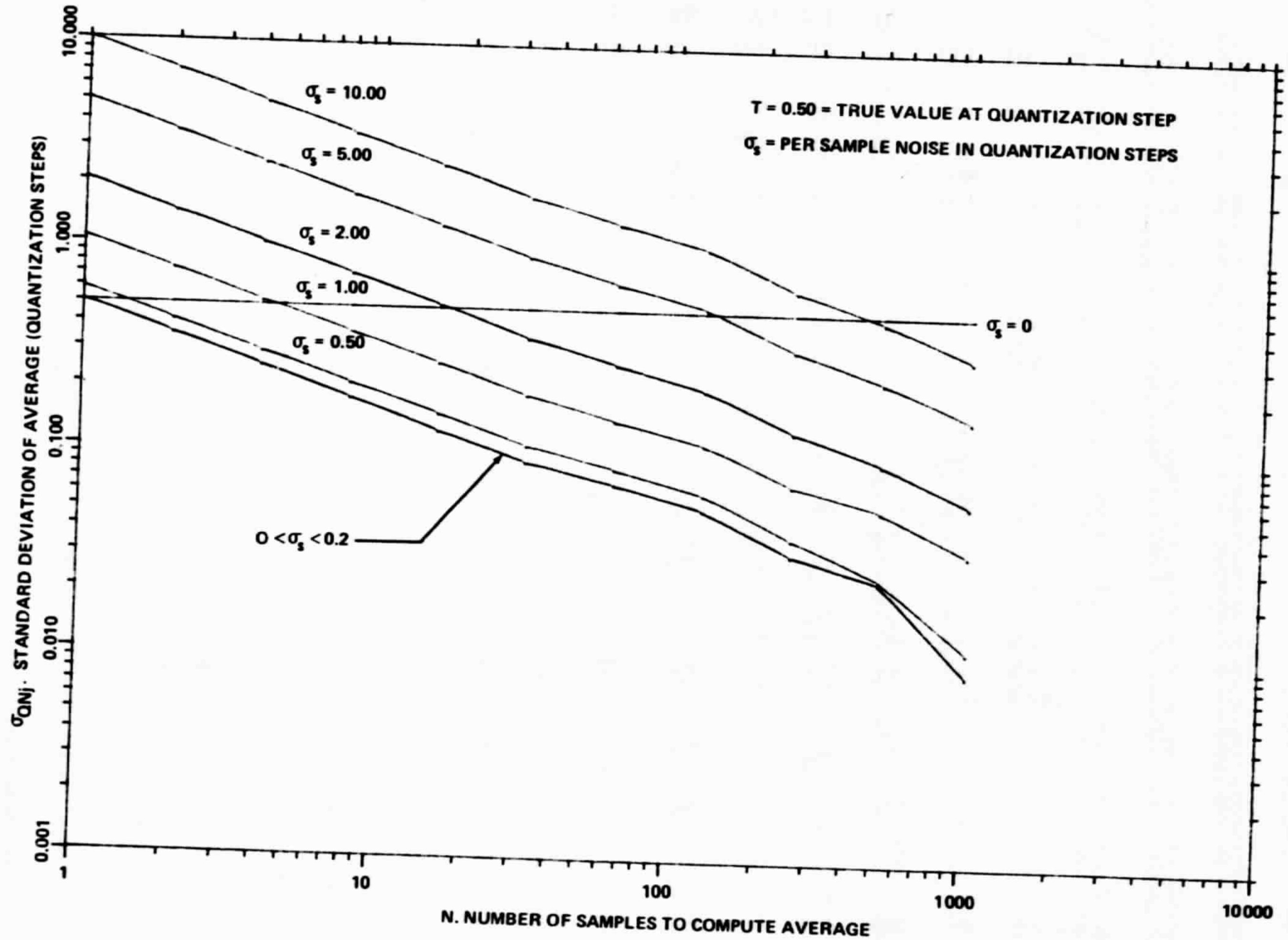

Figure C-4. Standard Deviation of the Average as a Function of the Number of Quantized Samples Used to Compute the Average for a True Signal Value $T_j = 0.5$

APPENDIX D

σ_{QN} AS A FUNCTION OF σ_s FOR VERY LARGE N ($N \to \infty$)

Let the true value T of an analog signal be defined by

$$T = Q + x, \qquad ½ < x < ½ \tag{D-1}$$

This true signal is corrupted by noise (with zero mean and variance σ_s) and quantized. This appendix will show what the variance of the average is about the true signal for averages based upon an infinite number of quantized samples.

Define a quantizing function G(S) for signal S by

$$G(S) = Q + i \text{ for } Q + i - ½ < S < Q + i + ½$$

where

$$i = -\infty, \ldots, -N, \ldots, -1, 0, 1, \ldots, N, \ldots, \infty.$$

Then the average, $\overline{S}$, of an infinite number of quantized samples is given by

$$\overline{S} = E\left\{G(S)\right\}$$

$$= \int_{-\infty}^{\infty} G(S)\, p(S) dS$$

or

$$\overline{S} = Q + F(x) \tag{D-2}$$

where

$$F(x) = \lim_{N \to \infty} \sum_{i=-N}^{N} i \int_{Q+\left(\frac{2i-1}{2}\right)}^{Q+\frac{2i+1}{2}} P(S)\, dS \tag{D-3}$$

and

$$P(S) = \frac{1}{\sigma_s\sqrt{2\pi}} e^{-(S-T)^2/2\sigma_s^2} \tag{D-4}$$

Let

$$z = S - T \tag{D-5}$$

in Eqn. (D-3) and after rearranging and utilizing Eqn. (D-1), obtain

$$F(x) = \lim_{N\to\infty} \sum_{i=1}^{N} \int_{\left(\frac{2i-1}{2}\right)-x}^{\frac{(2i-1)}{2}+x} p(z)\,dz \tag{D-6}$$

where

$$p(z) = \frac{1}{\sigma_s\sqrt{2\pi}} e^{-z^2/2\sigma_s^2} \tag{D-7}$$

One notes that $F(0) = 0$, $F(\frac{1}{2}) = \frac{1}{2}$ and $F(-\frac{1}{2}) = -\frac{1}{2}$.

The variance V of the average about the true value T is given by

$$V = \int_{-1/2}^{1/2} (\bar{S} - T)^2\,dx$$

or

$$V = \int_{-1/2}^{1/2} (F(x) - x)^2\,dx \tag{D-8}$$

But $(F(x) - x)^2$ is an even function of x since

$$[F(x) - x]^2 = [F(-x) + x]^2 \tag{D-9}$$

Hence

$$V = 2 \int_{0}^{1/2} [F(x) - x]^2\,dx \tag{D-10}$$

$$\sigma_{Q\infty} = \sqrt{V} \tag{D-11}$$

This is the desired result, i.e., the variance of the quantized signal about the true value for an infinite number of samples.

APPENDIX E

COMPUTATION OF GAUSSIAN NOISE

The gaussian distribution p(y) is given by

$$p(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-a)^2/2\sigma^2} \tag{E-1}$$

where

y = The random variable

a = The mean of the random variable y

σ = The standard deviation of y

Let

$$a = 0$$

$$\sigma = 1. \tag{E-2}$$

Fix y_i, i = 1 to w + 1, such that there is equal probability of .5/w for each interval given by

$$0 = a \leq y_i < y < y_{i+1} \text{ for } i = 1 \text{ to } w$$

The y_i's may be computed numerically from the equation

$$\int_{y_i}^{y_{i+1}} p(y)dy = \frac{.5}{w} \text{ for } i = 1 \text{ to } w$$

where

$$y_1 = a = 0$$

Let

$$\overline{y}_i = \frac{y_i + y_{i+1}}{2} \text{ for } i = 1 \text{ to } w \tag{E-3}$$

Let U be a uniformly distributed random variable between 0 and 1. Then random noise η with a gaussian distribution of zero mean and standard deviation of σ_s may be computed as follows

$$i = \mathrm{Int}\,(Uw) + 1 \tag{E-4}$$

$$\eta = \sigma_s \bar{y}_i \,\mathrm{sgn}\,(U - .5) \tag{E-5}$$

where U was computed by utilizing a standard IBM program for uniform random number generation.

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